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An asymptotic analytical solution to the problem of two moving boundaries with fractional diffusion in one-dimensional drug release devices

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Abstract
We set up a one-dimensional mathematical model with a Caputo fractional operator of a drug released from a polymeric matrix that can be dissolved into a solvent. A two moving boundaries problem in fractional anomalous diffusion (in time) with order $\alpha \in (0, 1]$ under the assumption that the dissolving boundary can be dissolved slowly is presented in this paper. The two-parameter regular perturbation technique and Fourier and Laplace transform methods are used. A dimensionless asymptotic analytical solution is given in terms of the Wright function.

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1. Introduction

The research on the mathematical model of controlled release of drugs from a polymeric matrix has been attracting more attention in the past decades. It is a typical moving boundary problem or Stefan problem from the point of view of mathematics. A one moving boundary problem only has a diffusing boundary if the matrix cannot be dissolved while a two moving boundaries problem has both a dissolving boundary and a diffusing boundary because of the dissolved matrix. The appearance of the moving boundary leads to nonlinearity and there are very few exact solutions. The moving boundary problems with integer order have been widely studied by many scientists [1].

If the time of release is very long, the diffusion processes usually no longer follow the Gaussian theorem, and Fick's second law fails to describe the related transport behavior, so the fractional operators should be introduced [2–4]. In fact, more and more experience has proved that diffusion in a complex system cannot be described by Fick's law. Instead, anomalous diffusion is found in a wide diversity of systems, its hallmark being the nonlinear growth of the mean-squared displacement in the course of time. As a special porous medium, the matrix of the drug has a fractional dimension. Fractional calculus is a powerful tool for describing it [5, 6].

Taking into account the effect of ultralong diffusion in the polymeric matrices, introducing the fractional operators to the Fickian diffusion equation can describe the process more accurately. Liu and Xu [7] first introduced a time-fractional diffusion equation with a one moving boundary condition to a drug release process. An exact solution of a one-dimensional mathematical model was given in that paper. The results given by them coincided with the well-known Ritger–Peppas semi-empirical formula [8] in the controlled drug release system. Li *et al* [9] used the space-time-fractional diffusion equation to describe the process of a solute release from a polymeric matrix in which the initial concentration is greater than the solubility of the drug and gave the exact solution in terms of the Fox-H function. But the space-fractional derivative used in that paper is a Riesz derivative which would be inaccurate to model the diffusion process in a finite domain from the point of Lévy flights. To overcome this problem Li *et al* [10] used a Riemann–Liouville derivative and a Caputo derivative as space derivatives of equations, respectively. They gave the similarity solutions in terms of a generalized Wright function [11]. A comparison between the solutions corresponding to two types of fractional derivative was also obtained in that paper.

All the research mentioned above is based on the one moving boundary problem. In many cases, the polymer materials as drug matrices are the rate-limiting membrane. The research on pharmacokinetics in dissolved matrices is useful for designing and prediction of drug delivery systems. Furthermore, dissolved matrices based on polymer materials disappear after implantation, and this is an important advantage for use in patients. The dissolving boundary that dissolved slowly is based on the matrix of the drug, and it is a complex system which can be described more accurately by fractional operators. The dissolved rate is slower than the diffusion rate, so the kinetics of release is similar to the diffusion from non-dissolved matrices. The condition that the dissolved rate is faster than the diffusion rate is not considered in this paper.

In this paper, the problem of two moving boundaries with a Caputo fractional operator with order $\alpha \in (0, 1]$ is studied. We assume that the matrix can be dissolved slowly (controlled by the parameter η which will be mentioned in the paper) and the initial loading is greater than the solubility of the drug (controlled by the parameter ε which will be mentioned in the paper). The technique of two-parameter regular perturbation and Fourier and Laplace transform methods are used. An asymptotic analytical solution is given in terms of the Wright function. Some discussion is given at the end of the paper.

2. Mathematical model and governing equations

In order to solve the problem of two moving boundaries with fractional diffusion, the mathematical model and the assumptions are given as follows:

- The polymer as a matrix can be dissolved slowly and is a one-dimensional slab;
- a perfect sink condition is assumed;
- the diffusivity D of the drug in the matrix is constant;
- the initial concentration C_0 of the drug is much greater than the solubility C_s of the drug, i.e. $C_0 \gg C_s$.

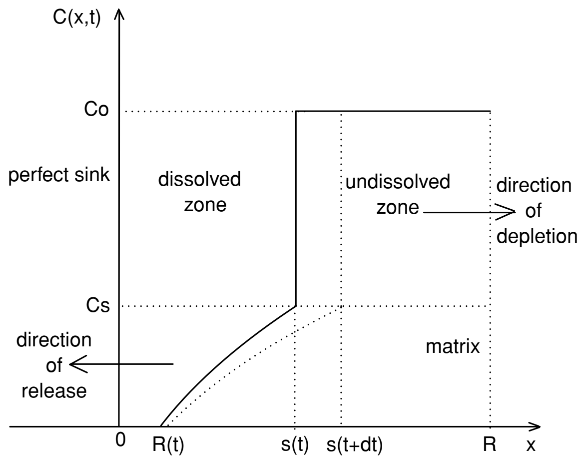

Figure 1. Profile of concentration.

The concentration profile at time t is shown in figure 1. We generalize Fick's law to describe this diffusion by using the tool of fractional calculus. The governing equation and conditions are as follows:

$$ {}_0^C D_t^{\alpha} C(x,t) = D\frac{\partial^2 C}{\partial x^2} \qquad (R(t) < x < S(t), 0 < \alpha \leqslant 1) \tag{1} $$

$$ C(x,t) = 0 \qquad (x = R(t)), \tag{2} $$

$$ C(x,t) = C_s \qquad (x = S(t)), \tag{3} $$

$$ (C_0 - C_s)\, {}_0^C D_t^{\alpha} S(t) = D\frac{\partial C}{\partial x}\bigg|_{x=S(t)} \quad (t > 0), \tag{4} $$

$$ R(t) = S(t) = 0 \qquad (t = 0). \tag{5} $$

Here $C(x,t)$ is the concentration of drug in the matrix. $S(t)$ is the position of the diffusing boundary at time t. $R(t)$ is the position of the dissolving boundary at time t. Condition (2) is the perfect sink condition at $x = R(t)$. Equation (4) is the mass balance equation at the diffusion interface like energy conservation in heat transfer with phase transition, which is known as the 'Stefan condition' [1] which is also used in the paper [7]. The Caputo fractional derivative and integral operators are respectively defined in reference [12]:

$$ {}_0^C D_t^{\alpha} f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,\mathrm{d}\tau\,(n-1 < \alpha < n); \tag{6} $$

$$ {}_0^C D_t^{-\beta} f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau \quad (0 < \beta < 1), \tag{7} $$

where $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\mathrm{d}t$ is the Gamma function.

The properties of the Caputo fractional derivatives can be found in [12, 13]. Three important properties used in this paper are the following.

Property 1

$$ {}_0^C D_t^{\alpha} t^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}, \qquad \alpha > 0, \quad \mu > -1. \tag{8} $$

Property 2

$$ {}_0^C D_t^\alpha A = 0, \qquad \alpha > 0, \tag{9}$$

where A is a constant.

Property 3

$$\mathcal{L}\left\{{}_0^C D_t^\alpha f(t); p\right\} = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha \leqslant n. \tag{10}$$

Property 3 is the Laplace transform of the Caputo derivative.

3. The asymptotic solution of the problem

The equation and conditions can be considerably simplified if we introduce the following dimensionless variables:

$$\theta(x,t) = \frac{C(x,t)}{C_s}, \qquad t^* = \left(\frac{D}{L^2}\right)^{\frac{1}{\alpha}} t, \qquad \varepsilon = \frac{C_s}{C_0},$$
$$R^*(t^*) = \frac{R(t)}{L}, \qquad S^*(t^*) = \frac{S(t)}{L}, \qquad x^* = \frac{x}{L},$$

where $\varepsilon \sim o(1)$ from the fourth assumption is the ratio of solubility C_s and initial concentration C_0 and L is the lengthscale.

For convenience, the mark '$*$' will be omitted in the rest of the paper and equations (1)–(5) become

$$ {}_0^C D_t^\alpha \theta(x,t) = \frac{\partial^2\theta}{\partial x^2} \qquad (R(t) < x < S(t), 0 < \alpha \leqslant 1), \tag{11}$$

$$\theta(x,t) = 0 \qquad (x = R(t)), \tag{12}$$

$$\theta(x,t) = 1 \qquad (x = S(t)), \tag{13}$$

$$(\varepsilon^{-1} - 1)\, {}_0^C D_t^\alpha S(t) = \left.\frac{\partial\theta}{\partial x}\right|_{x=S(t)} \qquad (t > 0), \tag{14}$$

$$R(t) = S(t) = 0 \qquad (t = 0). \tag{15}$$

We introduce the new dimensionless independent space-time variables:

$$y = x - R(t), \qquad X(t) = S(t) - R(t). \tag{16}$$

From the first assumption, we have

$$R(t) = \eta t, \qquad \eta \sim o(1), \tag{17}$$

where η means the dimensionless moving velocity of the dissolving boundary.

Inserting (16) and (17) into ${}_0^C D_t^\alpha \theta(x,t)$, we obtain

$$\begin{aligned} {}_0^C D_t^\alpha \theta(x,t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial\theta(x,\tau)}{\partial\tau}\, \mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \left[\frac{\partial\theta(y,\tau)}{\partial\tau} + \frac{\partial\theta(y,\tau)}{\partial y}\frac{\partial y}{\partial\tau}\right] \mathrm{d}\tau \\ &= {}_0^C D_t^\alpha \theta(y,t) - \eta \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial\theta(y,\tau)}{\partial y}\, \mathrm{d}\tau \\ &= {}_0^C D_t^\alpha \theta(y,t) - \eta \frac{\partial}{\partial y}\, {}_0^C D_t^{\alpha-1} \theta(y,t). \end{aligned}$$

Considering the new space-time variables, equations (11)–(15) become

$$ {}_0^C D_t^\alpha \theta(y,t) - \eta \frac{\partial}{\partial y} {}_0^C D_t^{\alpha-1} \theta(y,t) = \frac{\partial^2 \theta(y,t)}{\partial y^2} \qquad (0 < y < X(t)), \tag{18} $$

$$ \theta(y,t) = 0 \qquad (y = 0), \tag{19} $$

$$ \theta(y,t) = 1 \qquad (y = X(t)), \tag{20} $$

$$ (\varepsilon^{-1} - 1)\left[{}_0^C D_t^\alpha X(t) + \eta \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \right] = \left.\frac{\partial \theta}{\partial y}\right|_{y=X(t)} \qquad (t > 0), \tag{21} $$

$$ X(t) = 0 \qquad (t = 0). \tag{22} $$

The equations have the form of a one moving boundary problem after introducing the new variables.

Because the dimensionless parameters η and ε are smaller than one, we use the two-parameter regular perturbation method to solve the equations. We assume that the solutions of the equations can be given as follows:

$$ \theta(y,t;\eta,\varepsilon) = \theta_0(y,t) + \theta_1(y,t)\eta + \theta_2(y,t)\epsilon + \cdots, \tag{23} $$

$$ X(t;\eta,\varepsilon) = X_0(t) + X_1(t)\eta + X_2(t)\varepsilon + \cdots. \tag{24} $$

Substituting (23) and (24) into (18)–(22), we arrive at

$$ {}_0^C D_t^\alpha \theta_0(y,t) + \eta\, {}_0^C D_t^\alpha \theta_1(y,t) + \varepsilon\, {}_0^C D_t^\alpha \theta_2(y,t) - \eta \frac{\partial}{\partial y} {}_0^C D_t^{\alpha-1} \theta_0(y,t) \cdots $$
$$ = \frac{\partial^2 \theta_0}{\partial y^2} + \eta \frac{\partial^2 \theta_1}{\partial y^2} + \varepsilon \frac{\partial^2 \theta_2}{\partial y^2} \cdots, \tag{25} $$

$$ \theta_0(y,t) + \eta\theta_1(y,t) + \varepsilon\theta_2(y,t) \cdots = 0 \quad (y = 0), \tag{26} $$

$$ \theta_0(y,t) + \eta\theta_1(y,t) + \varepsilon\theta_2(y,t) \cdots = 1 \quad (y = X(t)), \tag{27} $$

$$ (1-\varepsilon)\left[{}_0^C D_t^\alpha X_0(t) + \eta\, {}_0^C D_t^\alpha X_1(t) + \eta \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \cdots \right] $$
$$ = \varepsilon \left.\left[\frac{\partial \theta_0}{\partial x} + \eta \frac{\partial \theta_1}{\partial x} + \varepsilon \frac{\partial \theta_2}{\partial x} + \cdots \right]\right|_{x=X(t)}, \tag{28} $$

$$ X_0(t) + \eta X_1(t) + \varepsilon X_2(y,t) \cdots = 0 \quad (t = 0). \tag{29} $$

By comparison with the same orders of $\eta^n \varepsilon^m$ $(n = 0, 1, \ldots; m = 0, 1, \ldots)$ of both sides of equations (25)–(29), we can obtain the perturbation equations of different orders. The zero-order equations corresponding to the term $\eta^0\varepsilon^0$ are as follows:

$$ {}_0^C D_t^\alpha \theta_0(y,t) = \frac{\partial^2 \theta_0(y,t)}{\partial y^2} \qquad (0 < y < X(t)), \tag{30} $$

$$ \theta_0(y,t) = 0 \qquad (y = 0), \tag{31} $$

$$ \theta_0(y,t) = 1 \qquad (y = X(t)), \tag{32} $$

$$ {}_0^C D_t^\alpha X_0(t) = 0 \qquad (t > 0), \tag{33} $$

$$ X_0(t) = 0 \qquad (t = 0). \tag{34} $$

The one-order equations corresponding to the terms $\eta^1\varepsilon^0$ and $\eta^0\varepsilon^1$ are

$$ {}_0^C D_t^{\alpha}\,\theta_1(y,t) - \frac{\partial}{\partial y}\,{}_0^C D_t^{\alpha-1}\,\theta_0(y,t) = \frac{\partial^2\theta_1(y,t)}{\partial y^2}(0 < y < X(t)), \tag{35} $$

$$ \theta_1(y,t) = 0 \qquad (y = 0), \tag{36} $$

$$ \theta_1(y,t) = 0 \qquad (y = X(t)), \tag{37} $$

$$ {}_0^C D_t^{\alpha}\,X_1(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} = 0 \qquad (t > 0), \tag{38} $$

$$ X_1(t) = 0 \qquad (t = 0), \tag{39} $$

and

$$ {}_0^C D_t^{\alpha}\,\theta_2(y,t) = \frac{\partial^2\theta_2(y,t)}{\partial y^2}(0 < y < X(t)), \tag{40} $$

$$ \theta_2(y,t) = 0 \qquad (y = 0), \tag{41} $$

$$ \theta_2(y,t) = 0 \qquad (y = X(t)), \tag{42} $$

$$ {}_0^C D_t^{\alpha}\,X_2(t) - {}_0^C D_t^{\alpha}\,X_0(t) = \left.\frac{\partial\theta_0}{\partial y}\right|_{y=X(t)} \qquad (t > 0), \tag{43} $$

$$ X_2(t) = 0 \qquad (t = 0), \tag{44} $$

respectively.

Using the method used in paper [7] and property 2, the solutions to equations (30)–(34) are given as follows:

$$ \theta_0(y,t) = H_0\left[1 - W\left(-\frac{y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right)\right], \tag{45} $$

$$ X_0(t) = 0, \tag{46} $$

where H_0 is a constant to be determined in the following part of the paper and $W(z;\alpha,\beta)$ is the Wright function defined as [14]

$$ W(z;\alpha,\beta) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)}, \qquad \alpha > -1, \quad \beta\in\mathbb{C}. \tag{47} $$

We insert (45) into $\frac{\partial}{\partial y}\,{}_0^C D_t^{\alpha-1}\,\theta_0(y,t)$ and obtain

$$ \begin{aligned} \frac{\partial}{\partial y}\,{}_0^C D_t^{\alpha-1}\,\theta_0(y,t) &= -H_0\frac{\partial}{\partial y}\,{}_0^C D_t^{\alpha-1}\,W\left(-\frac{y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right) \\ &= -H_0\frac{\partial}{\partial y}\,{}_0^C D_t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(yt^{-\alpha/2})^k}{k!\Gamma\left(-\frac{\alpha}{2}k+1\right)} \\ &= -H_0\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(-\frac{\alpha}{2}k+1\right)}\frac{\partial}{\partial y}y^k\,{}_0^C D_t^{\alpha-1}\,t^{-\frac{\alpha}{2}k} \\ &= H_0\sum_{k=1}^{\infty}\frac{\left(-yt^{-\frac{\alpha}{2}}\right)^{k-1}t^{1-\frac{3\alpha}{2}}}{(k-1)!\Gamma\left(-\frac{\alpha}{2}(k-1)+2-\frac{3\alpha}{2}\right)} \\ &= H_0 t^{1-\frac{3\alpha}{2}}\,W\left(\frac{-y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 2-\frac{3\alpha}{2}\right). \end{aligned} \tag{48} $$

Substituting (48) into equation (35), we have

$$ {}_0^C D_t^\alpha \theta_1(y,t) - H_0 t^{1-\frac{3\alpha}{2}} W\left(\frac{-y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 2-\frac{3\alpha}{2}\right) = \frac{\partial^2 \theta_1(y,t)}{\partial y^2}. \tag{49} $$

Taking the Laplace transform of (49) in the time domain yields

$$ p^\alpha \widetilde{\theta}_1(y,p) - H_0 p^{\frac{3\alpha}{2}-2} \exp\left(-y p^{\frac{\alpha}{2}}\right) = \frac{\partial^2 \widetilde{\theta}_1}{\partial y^2}. \tag{50} $$

Then taking the Fourier transform of (50) in the space domain we arrive at

$$ (p^\alpha + k^2)\widehat{\widetilde{\theta}}_1(k,p) = 2H_0 \frac{p^{2\alpha-2}}{p^\alpha + k^2}, \tag{51} $$

where p and k are variables of Laplace and Fourier transforms, respectively.

From (51) we get the solution

$$ \widehat{\widetilde{\theta}}_1(k,p) = \frac{2H_0 p^{2\alpha-2}}{(p^\alpha + k^2)^2}. \tag{52} $$

Using the inverse Fourier and Laplace transforms, we have

$$ \theta_1(y,t) = \frac{\pi H_0}{2} y t^{1-\alpha} W\left(\frac{-y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 2-\alpha\right) + \frac{\pi H_0}{2} t^{1-\alpha/2} W\left(\frac{-y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 2-\frac{\alpha}{2}\right). \tag{53} $$

For $X_1(t)$, we have the equation

$$ {}_0^C D_t^\alpha X_1(t) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} = 0. \tag{54} $$

By using property 1, the solution is

$$ X_1(t) = -t. \tag{55} $$

To solve equation (40) with conditions (41)–(44), using the same method used to obtain θ_0, we have

$$ \theta_2(y,t) = H_1\left[1 - W\left(-\frac{y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right)\right], \tag{56} $$

where H_1 is also a constant to be determined in the following part of the paper.

Because θ_0 was given in (45), we have

$$ \frac{\partial \theta_0}{\partial y} = H_0 t^{-\alpha/2} W\left(\frac{-y}{t^{\alpha/2}}; -\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right). \tag{57} $$

Inserting (57) into equation (43), we obtain

$$ {}_0^C D_t^\alpha X_2(t) - {}_0^C D_t^\alpha X_0(t) = \frac{H_0 t^{-\alpha/2}}{\Gamma(1-\alpha/2)}, \tag{58} $$

and its solution is

$$ X_2(t) = \frac{H_0 t^{\alpha/2}}{\Gamma(1+\alpha/2)}. \tag{59} $$

So the asymptotic solutions to the governing equations are

$$ \theta(y,t;\eta,\varepsilon) = \theta_0(y,t) + \theta_1(y,t)\eta + \theta_2(y,t)\varepsilon + \cdots, \tag{60} $$

$$ X(t;\eta,\varepsilon) = X_0(t) + X_1(t)\eta + X_2(t)\varepsilon + \cdots, \tag{61} $$

where $\theta_0, \theta_1, \theta_2 \ldots$ and $X_0, X_1, X_2 \ldots$ have been obtained.

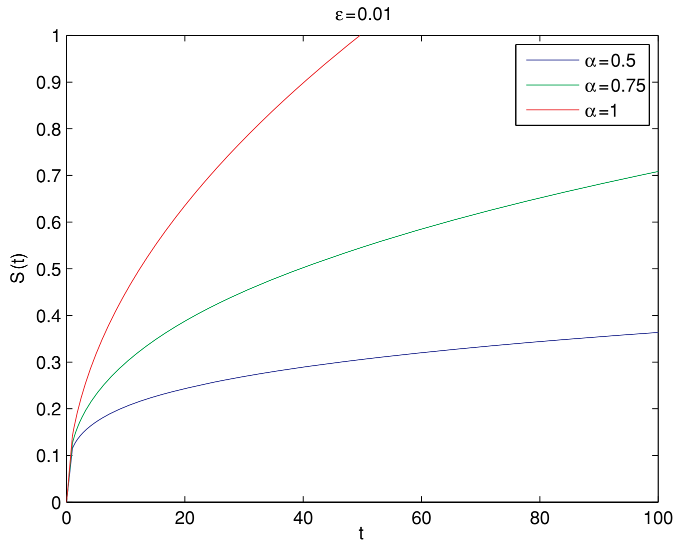

Figure 2. The dimensionless diffusion interface $S(t)$ versus dimensionless time t with different α.

4. Discussion

Considering the special case ($\eta = 0$), the problem of two moving boundaries becomes the one moving boundary problem discussed in [7]. Using the same method, we obtain

$$H = \frac{1}{1 - W\left(-q; -\frac{\alpha}{2}, 1\right)},$$
$$H = \frac{q\Gamma\left(1+\frac{\alpha}{2}\right)\left(\varepsilon^{-1}-1\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)W\left(-q; -\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right)}, \tag{62}$$

where H and q are determined by the equations above, and q is the dimensionless velocity of the diffusion interface

$$S(t) = q \cdot t^{\alpha/2}. \tag{63}$$

Figures 2 and 3 show how the rate of difusion interface varies with t. From figure 2, the diffusion interface will run faster if α increases. From figure 3, the diffusion interface will run faster if ε increases. The parameter α can be controlled by using different polymer matrices, while the parameter ε can be controlled by using different loading drugs. We can control drug release by varying the two parameters.

Expanding H in a power series of ε, we obtain

$$H = H_0 + H_1\varepsilon \cdots. \tag{64}$$

So the constants H_0, $H_1 \ldots$ are determined, and so we can know the asymptotic solution from (61):

$$H\varepsilon = H_0\varepsilon + H_1\varepsilon^2 + \cdots$$
$$= \frac{q\Gamma\left(1+\frac{\alpha}{2}\right)(1-\varepsilon)}{\Gamma\left(1-\frac{\alpha}{2}\right)W\left(-q; -\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right)}, \tag{65}$$

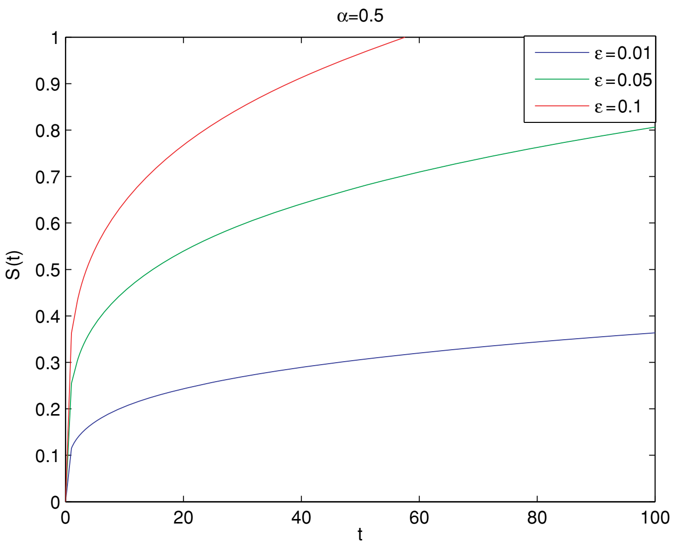

Figure 3. The dimensionless diffusion interface $S(t)$ versus dimensionless time t with different ε.

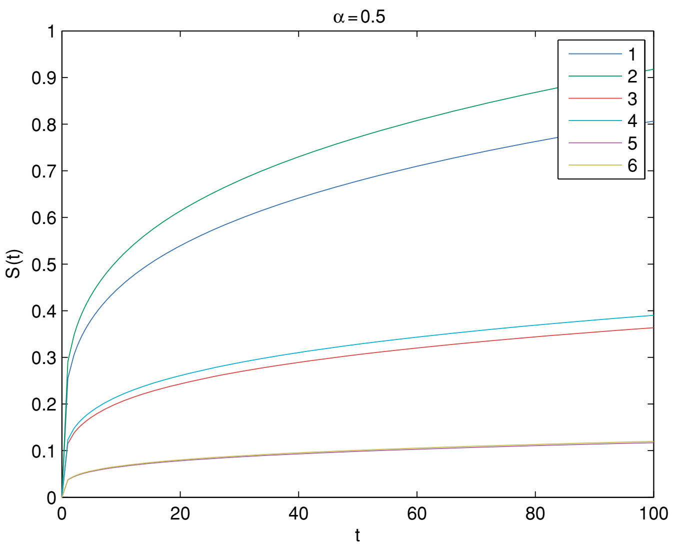

Figure 4. The comparison between the exact solution and the asymptotic solution (when $\alpha = 0.5$). Curves 1, 3, 5 correspond to the exact solution when $\varepsilon = (0.05, 0.01, 0.001)$ and curves 2, 4, 6 correspond to the asymptotic solution when $\varepsilon = (0.05, 0.01, 0.001)$, respectively.

$$S(t) = X(t) \approx \frac{H_0 \varepsilon}{\Gamma(1+\alpha/2)} \cdot t^{\alpha/2}$$
$$\approx \frac{q(1-\varepsilon)}{\Gamma\left(1-\frac{\alpha}{2}\right) W\left(-q; -\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right)} \cdot t^{\alpha/2} \quad (\varepsilon = o(1)). \tag{66}$$

Figure 4 shows the comparison between the exact solution and the asymptotic solution. When ε is determined, the asymptotic solution curve runs faster than the exact solution curve, and the difference between the exact solution and the asymptotic solution become smaller when ε decreases.

Considering the dissolving boundary condition, we have the position of two moving boundaries:

$$R(t) = \eta t, \tag{67}$$

$$\begin{aligned} S(t) &= X(t) + R(t) \\ &= X_0(t) + X_1(t)\eta + X_2(t)\varepsilon + \eta t + \cdots \\ &= \frac{H_0 t^{\alpha/2}}{\Gamma(1+\alpha/2)}\varepsilon + \cdots. \end{aligned} \tag{68}$$

Because the dissolved rate is very slow, it cannot obviously influence the diffusion interface and the rate of diffusion interface would be the main factor of drug release. Although the smaller ε can make the exact solution and asymptotic solution closer, it makes the diffusion interface run slower, too. If the dissolved interface runs faster than the diffusion interface, the dissolved rate of the matrix would be the main factor of drug release and another method should be used.

5. Conclusion

For moving boundary problems, very few analytical solutions are available in closed form [1]. Some scientists have researched the one moving boundary with fractional operators and obtained some exact solutions. In this paper, we set up the mathematical model of a two moving boundaries problem in fractional diffusion and an asymptotic solution is given by means of the two-parameter regular perturbation technique. But our model is according to the assumption that the dissolving boundary is dissolved very slowly. If the dissolving boundary moves fast or the matrix can swell, our solution cannot be used and a new method must be developed to solve the new problem.

Fractional calculus is a powerful tool for researching the ultralong diffusion process. Our research can be a type of method to obtain the asymptotic solution. The perturbation has been used to solve the moving boundary problem with integer order [15, 16] and we prove that it can also be used in the fractional moving boundary problem.

Acknowledgments

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